Aberrations in Holography

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Abstract. The Seidel aberrations are described as they apply to holography. Methods of recognition of an aberrated holographic reconstruction are described, as well as a recognition of the type of aberration. Experimental and theoretical strategies to minimize aberrations are discussed, including geometric considerations in the recording and reconstruction of holograms. Aberrations due to the recording of a hologram with one wavelength and reconstructing of it in another is also examined.

1. Introduction

An aberration in an imaging system is defined as any distortion of an image due to imperfections in the imaging system. In particular, the monochromatic aberrations, known as the Seidel aberrations, are aberrations irrespective of dispersion in the optical imaging system. In a wave picture, the Gaussian image point is considered to be the center of a spherical wavefront, known as the reference sphere, with the distance from the Gaussian image point to the exit pupil as the radius. The phase mismatch between actual wavefront and the ideal reference sphere is then the aberration. The parameters of the aberration function are the height of a point on the wavefront from the axis, or principal ray, denoted by r, the azimuthal angle of a point on the wavefront, denoted by ϕ and the height of the Gaussian image point from the axis, denoted by σ

Eq.1 $$W = {}_{o}C_{40}(r^4) + {}_{1}C_{31}(\sigma)(r^3)\cos(\phi) + {}_{2}C_{22}(\sigma)^2 (r^2)\cos^2(\phi) + {}_{2}C_{20}(\sigma)^2 (r^2) + {}_{3}C_{11}(\sigma)^3 (r)\cos(\phi)$$

The aberrations described by the five terms in the aberration function above are known as the five Seidel aberrations of: spherical aberration, coma, astigmatism, petzval curvature and distortion. The first three terms lead to a blurring of the image point and the next two lead to a distortion of the image.

2. Analysis of hologram formation and aberrations

The wave picture may be used to create an analogue of the Seidel aberrations in holography by determining the variation of a wavefront generated by the reconstruction of a hologram to an ideal wavefront that should be generated by the same hologram. It will be found that an aberration function, W, can be derived under these conditions that predict the same parameters as may be predicted by using the lens aberration function. In particular, the parameters of the position of an image, it's lateral, angular and longitudinal magnification and its five Seidel aberrations may be predicted. The basic theory assumes a point source as an object and so does not take into account extended objects such as planes.

The analysis of the construction and reconstruction of an image follows Meier [1] and

Champagne [2]. It assumes a point source as the object and a point source for the reference (Fig 1). The analysis then assumes a different point source for the origin of the reconstruction source. From the analysis, an image point (X_i, Y_i, Z_i) is derived using the coordinates of the object point (x_o, y_o, z_o), the original reference point (x_r, y_r, z_r) and the reconstruction point (x_c, y_c, z_c). Seidel aberrations are then determined by the phase mismatch between the image point as reconstructed by the original reference point source, the ideal image point, and the image point as reconstructed from a different point. The hologram itself is positioned in the x-y plane in this coordinate system with z=0. The analysis takes into account the fact that both the original reference and the reconstruction source may be collimated, i.e. the point of origin of these beams may be infinity. In this situation it is not possible to determine the exact coordinates of the original reference wave and the reconstruction wave and so only the directions of the reference and the reconstruction beams are the parameters on which the aberration is based. Furthermore, changes in wavelength between recording and reconstruction are also incorporated. Thus, if the recording wavelength is λo and the reconstruction wavelength is λc, then the ratio of the recording wavelength to the reconstruction wavelength is $\mu = (\lambda c / \lambda o)$. Both Meier and Champagne essentially use the same techniques, but Meier analyses the paraxial case and Champagne the non-paraxial case.

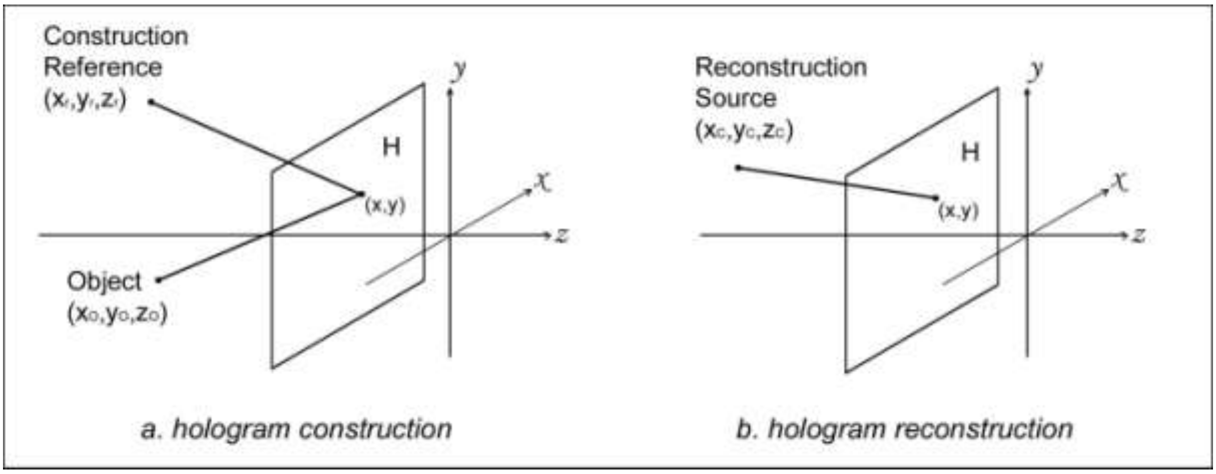

Fig 1. The geometry of recording and reconstruction of a hologram in this analysis.

The reconstructed phase wavefront emerging from the hologram at (x,y) in the hologram plane is then calculated as the sum of the recorded phases, ϕr and ϕo and the phase of the reconstructed wave, ϕc. Thus the total phase at the hologram, on reconstruction, is

Eq.2 $$\Phi = \phi c +/- \phi r +/- \phi o$$

where the signs are dependent on whether or not the reconstruction was carried out with the original reference beam or its conjugate and whether or not the desired image is the original (virtual) image or its (real) conjugate. The general expression for the phase of a spherical wave, from a general point (x_g, y_g, z_g) to a point (x, y) is given exactly by:

Eq.3 $$\Phi = 2\pi/\lambda\{[(x - x_g)^2 + (y-y_g)^2 + z_g{}^2]^{1/2} - [x_g{}^2 + y_g{}^2 + z_g{}^2]^{1/2}\}$$

If now Eq.3 is substituted into Eq.2 with the specific coordinates of the object, reference and reconstruction coordinates replacing the general coordinates, an exact expression is obtained for the phase Φ at the hologram plane H. However, a computationally simpler expression may be derived by expanding the exact phase expansion in a Taylor expansion about the z coordinates, assuming that $z_g{}^2$

> $x_g^2 + y_g^2$. To determine the position of the image point and the magnifications of the image, a first order expansion is sufficient. Under these circumstances, the phase of the reconstructed wave, i.e. the reconstructed wavefront, to first order, is:

$$\Phi^{(1)} = (\pi/\lambda c)\{(x^2 + y^2)(1/z_c + \mu/z_o - \mu/z_r) - 2x(x_c/z_c + \mu x_o/z_o - \mu x_r/z_r) - 2y(y_c/z_c + \mu y_o/z_o) - \mu y_r/z_r\}$$

This wave, ideally reconstructed, should be a spherical wavefront converging towards an image point at (Xi, Yi, Zi). By comparison with the ideal spherical wavefront, expressions are derived for the image position (here the Yi expression is not expressly stated but is exactly the same as the Xi expression except to replace all the x values with the appropriate y values):

Eq.4 $$Zi = z_o z_c z_r/(z_o z_r + \mu z_c z_r - \mu z_c z_o)$$

Eq.5 $$Xi = (x_c z_o z_r + \mu x_o z_c z_r - \mu x_r z_c z_o)/(z_o z_r + \mu z_c z_r - \mu z_c z_o)$$

It can be seen that if (x_c, y_c, z_c) is equal to (x_r, y_r, z_r), i.e. the reconstruction point is identical to the reference point, and $\mu = 1$, i.e. the hologram is reconstructed with the same wavelength with which it was recorded, then $Zi = z_o$ and $Xi = x_o$. That is, the image coordinates are the same as the object coordinates and the image appears in the same position as the original object with no magnification or aberrations.

3. Resolution

The resolution of a hologram is often brought up and compared against the resolution of electronic displays. The validity of such a comparison is doubtful, since a hologram is a diffractive effect whereas in an electronic display, the resolution of the display is determine by the pixel density. However, a plausible figure may be given for the resolution of a hologram by determining the variation of the size of the image point with respect to the reconstruction source size. While this calculation may be carried out more rigorously using the Van Cittert–Zernike theorem and so calculating the degree of spatial coherence of the source, a simpler back-of-an-envelope approach may be used. The variation of size of an image with respect to the variation in the source size, both laterally and longitudinally, may be given by $\partial X_i/\partial x_c$ and $\partial Z_i/\partial z_c$. To this end, from Eq.4 and Eq.5 we derive

Eq.6a $$\partial X_i/\partial x_c = (z_o z_r)/(z_o z_r + \mu(z_o z_r - z_o z_c))$$

Eq.6b $$\partial Z_i/\partial z_c = (z_o z_r)/(z_o z_r + \mu(z_c z_r - z_c z_o))$$

It can be seen that in Eq.6a, the lateral resolution, if the z coordinate of the reconstruction point is identical with that of the reference point, then the resolution of the image is equal to the size of the source, regardless of any change in wavelength. However, for the longitudinal case, Eq.6b, this is not so. If the z coordinates are now identical depends on both the z coordinate of the object position and the change in wavelength.

4. The Holographic Model

In order to show a correlation between holography and conventional lenses, an equivalent "focal length", f_R for the hologram may be derived from the reconstructed phase wavefront. This is allowable insofar as the reconstructed phase wavefront is also a converging spherical wavefront and maybe thought of as the "reference sphere". Thus, for the object beam coordinate, z_o, and the equivalent image position Zi, under the assumption of a spherical wavefront, we may construct

$$(1/z_o) + (1/Zi) = 1/f_R$$
$$= 1/z_o +/- (\mu)(1/z_o + 1/z_r)$$

by comparing the standard lens equation $1/s_i + 1/s_o = 1/f$
and also Champagne's derivation of image distance $1/Rf = 1/Rc +/- (\mu)(1/Ro + 1/Rr)$

The +/- refers to the either the virtual image or the real image.

If the reference and/or the reconstruction beam is collimated, i.e. z_r and/or $z_c = \infty$, then there is no unique position of the reference and reconstruction beams. In this case, the ratio of the x coordinate to the z coordinate may be used to determine the beam direction with reference to the recording surface normal, as opposed to a unique position. These ratios are the sin of the angles of the beams, thus, the image is defined by its direction:

$$Xi/Zi = x_c/z_c +/- \mu(x_o/z_o - x_r/z_r)$$

Which leads to,

$$\text{Sin}\alpha_\iota = \text{Sin}\alpha_c + \mu(\sin\alpha_o - \sin\alpha_r)$$

Where α_ι is the direction of the image position, α_o is the direction of the object position, α_r is the original reference direction and α_c is the reconstruction beam direction. The Meier case is concerned with the paraxial case and poses no boundaries on the values of Xi and Yi by use of the paraxial approximation. While the non-paraxial case analysed by Champagne has the restriction that, for image formation

$$(Xi/Zi)^2 + (Yi/Zi)^2 < 1$$

If this condition is not satisfied, then the image field becomes evanescent and the hologram cannot be reconstructed.

5. Magnification in holography

There are three different kinds of magnification that must be taken into account: lateral magnification, M_T, longitudinal magnification, M_L, and angular magnification, M_A. Lateral magnification, M_T, is the increase in the lateral size of the image, i.e. its increase in the xy direction. This is given by the ratio of the object's original distance to the image's distance as the original distance varies. This is

$$M_T = dX_i/dx_o = 1/(1 +/- z_o/\mu z_c - z_o/z_r) \quad \text{(Meyer; paraxial)}$$
$$M_T = (\cos\alpha_o/\cos\alpha_i)\{1/(1 +/- z_o/\mu z_c - z_o/z_r)\} \quad \text{(Champagne; non-paraxial)}$$

Three interesting facts emerge:

1. If there is no change in wavelength between the recording wavelength and the reconstruction wavelength ($\mu = 1$), and the reconstruction distance is the same as the reference distance ($z_c = z_r$), then the magnification is unity, i.e. there is no magnification.
2. If the reconstruction beam is collimated ($z_c = \infty$), then the μ factor drops out. That is, the magnification of an image by change in wavelength occurs only for diverging or converging reconstruction beams. A collimated reconstruction beam will result in unit magnification.
3. Conventional magnification by a lens of focal length f is given by $M = f/(z_o - f)$. If the equivalent holographic focal length f_R, as derived above, is used as f in this expression, then the conventional magnification is different from the holographic magnification by a factor $1/\mu$. This shows the limitation of the lens metaphor for the hologram; the lens and holographic

magnifications are only equal if the reconstruction wavelength is identical to the recording wavelength. So, a hologram is the equivalent of a lens only insofar as there is equivalence in wavelengths.

Longitudinal magnification is a magnification along the axis of the hologram. That is, the variation of the image z coordinate, Z_i, to the original object's z coordinate, z_o, as z_o varies. This is:

$$M_L = dZ_i/dz_o = -(1/\mu)\{1/(1-z_o[(1/\mu z_c) + (1/_{zr})\}2 = -(1/\mu)M_T$$

Whereas in conventional imaging systems by lenses $M_L = -M_T{}^2$, the above shows that a holographic image is compressed along the axis by a factor of $(1/\mu)$ from conventional imaging systems [3]. Note also that if the reconstruction beam is collimated, then $M_T = 1$ (no lateral magnification), but M_L is still reduced by a factor of $(1/\mu)$. In other words, for a collimated reconstruction beam, there is longitudinal magnification with no transverse magnification. The angular magnification is given by the ratio of the angle subtended by the image to the angle subtended by the original object at the recording surface:

$M_A = d(X_i/Z_i)/d(x_o/z_o) = \mu$ (Meier; paraxial)
$M_A = \mu(\cos\alpha_o/\cos\alpha_i)$ (Champagne; nonparaxial)

Note that this magnification is independent of all the parameters of recording and reconstruction and depends only on the wavelength change. If there is no change in the wavelength, there is no angular magnification.

6. 3rd Order Seidel Aberrations in Holography

Seidel-equivalent aberrations for holography may be derived in the same manner as for lenses. The actual wavefront is compared against the ideal reference sphere and the mismatch between the wavefront and the reference sphere is taken as the aberration. However, in holography, the aberrated wavefront is calculated by carrying out a third order expansion of Eq.3, to determine the ideal, spherical wavefront, then substituting Eq.3 into Eq.2 and expanding to third order to obtain the third order expansion of the actual phase wavefront. The mismatch between the reference sphere and the third order expansion terms now gives the Seidel aberrations.

Transforming to a circular coordinate system, this gives for the Seidel aberration function:

W =	$2\pi/\lambda_c\, [(-1/8)\, \rho^4 S$	Spherical Aberration
	$+(1/2)\, \rho^3(C_x\cos\theta\ C_y\sin\theta)$	Coma
	$-(1/2)\, \rho^2(A_x\cos^2\theta + A_y\sin^2\theta + 2A_{xy}\cos\theta\sin\theta$	Astigmatism
	$+1(1/4)\, \rho^2 F$	Petzval Curvature
	$+(1/2)\, r(D_x\cos\theta + D_y\sin\theta)]$	Distortion

7. Discussion

The spherical aberration, S, is:

$$S = \mu[(\mu^2 - 1)(1/z_o{}^3 - 1/z_r{}^3) - (3\,\mu/z_c)(1/z_o{}^2 + 1/z_r{}^2) + 3(1/z_c{}^2 - (\mu^2/(z_o z_r))(1/z_o - 1/z_r) + 6(\mu/(z_o z_r z_c))]$$

If both z_r and z_c are infinite, i.e. both reference and reconstructions beams are collimated, then

$$S = \mu[\,\mu^2 - 1]/z_o{}^3$$

This becomes zero for $\mu = 1$. Thus, if the hologram is reconstructed with the same wavelength as the original reference and both reference and reconstruction beams are collimated, spherical aberration disappears. However, if either the reference or the reconstruction is not collimated, or reconstruction occurs with a different wavelength then spherical aberration will exist.

Coma, Cx, is:

$$Cx = (x_c/z_c)[(1/z_c^2) – (1/z_c - \mu/z_o + \mu/z_r)^2] – (\mu x_o/z_o)[(1/z_o^2) – (1/z_c - \mu/z_o + \mu/z_r)^2] + (\mu x_r/z_r)[1/z_r^2 – (1/z_c - \mu/z_o + \mu/zr)^2]$$

If both reference and reconstruction beams are collimated ($z_c = z_r = \infty$), then the ratios x_c/z_c and x_r/z_r are used, where the ratios are the tangents of angles of the beams with respect to the recording normal. Hence,

$$Cx = - (\mu/z_o^2)[\mu\tan\theta_c – (\mu^2 – 1)\tan\theta_o + \mu^2\tan\theta_r]$$

Coma can be made to disappear if both reference and reconstruction beams are collimated, the object is directly in front of the medium (resulting in θ_o =0 and so $\tan\theta_o = 0$), and $\tan\theta_c = -\mu\tan\theta_r$. Thus, if the c and r beams are collimated, the coma can be made to disappear if the tan of the reconstruction angle is modulated by the ratio of the recording wavelength to the reconstruction wavelength. If both these wavelengths are the same and the reconstruction angle is the conjugate of the recording angle, with both being collimated, coma can be made to disappear. In addition, if both wavelengths are the same, the condition on the object positions vanishes.

Astigmatism, Ax, is:

$$A_x = (x_c^2/z_c^3) – (\mu x_o^2/z_o^3) + (\mu x_r^2/z_r^3) – (x_c/z_c - \mu x_o/z_o + \mu x_r/z_r)^2(1/z_c - \mu/z_o + \mu/z_r)$$

For a collimated reference and reconstruction ($z_c = z_r = \infty$), the ratio x/z is taken and tangents of the various angles are used, as above. The astigmatism reduces to:

$$Ax = (\mu/z_o)[(\tan^2\theta_o)(\mu^2 – 1) - 2\mu \tan\theta_o (\tan\theta_c + \mu\tan\theta_r) + (\tan\theta_c + \mu\tan\theta r)^2]$$

Here the same three conditions that made coma disappear can also be used to make astigmatism disappear. That is, astigmatism disappears if $\tan\theta_c = -\mu\tan\theta_r$.

Petzval Curvature, F, is:

$$F = (x_c^2 + y_c^2)/z_c^2 - \mu(x_o^2 + y_o^2)/z_o^3 + \mu(x_r^2 + y_r^2)/zr^3 – (Xi^2 + Y_i^2)/Zi^3$$

The same conditions as those of astigmatism and coma will also make field curvature disappear. However, if the reconstruction beam is not collimated and/or μ is not equal to 1, then the image will be seen to bulge outward giving the image a “rolling” look.

Distortion, Dx, is:

$$Dx = \mu[(x_o/z_o)^3(\mu^2 – 1) –3\mu(x_o/z_o)^2(x_c/z_c+ \mu x_r/z_r) + 3(x_o/z_o)(x_c/z_c + \mu x_r/z_r)^2 – (x_r/z_r)^3(\mu^2 – 1) –3(x_o/z_o)(x_r/z_r)(x_c/z_c + \mu x_r/z_r) + x_o(y_o/z_o)^2(\mu^2 – 1)]$$

Once more, the same conditions will make distortion disappear.

8. Experimental Results

An planar image was recorded at 514nm with a collimated reference at 30 degrees to the plate normal at 13in from the recording plate.

The hologram was then reconstructed under a variety of beam geometries:
1. A collimated beam at various angles relative to the plate normal.
2. A range of diverging and converging beams. The direction of the reference was also changed by +/- 10 degrees from the original collimated beam direction.
3. The above two sets of experiments repeated with a wavelength of 633nm.

Table 1 gives the results for the first set of tests reconstructing at both 514nm (the original recording wavelength) and at 633nm. The angles are all relative to plate normal, with 30 degrees being the angle at which the hologram was recorded. The displacement refers to the displacement of the image in a plane parallel to the plane of the image. If a reference system is set up with the image reconstructed the x-y plane (z=0) and the z axis pointing away from the recording plate, the signs of the displacements are the signs of the x value. Thus, for a viewer standing behind the image, right is positive.

Table 2 and 3 give the displacement and twist of the image as a result of reconstructing the image with a diverging and converging beams. The divergence and convergence were accomplished by moving the spatial filter +/- 10 in from the focal position.

Angle	15	20	25	30	35	40	45
Displacement at 514	-3.0	-1.8	- 0.8	+ 0.3	+ 1.3	+ 2.3	+ 3.1
Displacement at 633		-2.8		- 1.25		+ 0.25	

Table 1. Lateral image displacement for reconstructions with collimated at various angles

θ	ϕ ~ -10 (divergent)		0 (collimated)		ϕ ~ +10 (convergent)	
	Image plane	twist	Image plane	twist	Image plane	twist
20o	14.5	+ 5.28o	13.13	-6.28o	11.56	-7.97o
30o	14.31	flat	13.06	flat	11.69	+ 1.86o
40o	14.25	-5.28o *	13.0	-1.72o	11.44	-4.43o

Table 2. Displacement and twist of the image for $\lambda = 514$. The θ figure gives the direction of the reconstruction beam.

θ	ϕ ~ -10 divergent		collimated		ϕ ~ +10 convergent	
	Image plane	twist	Image plane	twist	Image plane	twist
20o	11.31	-3.57o	10.31	-2.72o	9.5	-5.43o
30o	11.56	-1.86o	10.5	- .86o	9.5	-1.72o
40o	11.44	- .86o	10.63	- .86o	9.81	+ 1.72o

Table 3. Displacement and twist of the image for $\lambda = 633$. The θ figure gives the direction of the reconstruction beam.

9. Discussion of experimental data

From table 1, it can be seen that the lateral displacement follows the direction of the change in the (collimated) reference beam, being negative when the reconstruction beam is less than the recording angle and positive when it's greater. The displacement should be zero for the reference angle (30 degrees) when reconstructed at 514, but is measured at 0.3. This is due to the swelling caused by processing. Note that the reconstruction angle for the hologram recorded at 30 and reconstructed at 633 is 38, which is reflected in the results; the displacement for reconstruction at 633 is at a minimum at 40 degrees. Thus the displacement caused by altering the reconstruction angle by 50% at either wavelength causes an equal lateral displacement at both wavelengths.

In tables 2 and 3, the image plane refers to the distance between the image plane and the recording plate in the z direction (normal to the plate). Tables 2 and 3 both show that if the hologram is reconstructed with a divergence different to that of the recording divergence, there is a twist of the image and a normal (z direction) displacement. The amount of displacement also depends on the reconstruction wavelength. The object plane was recorded at 13.0 in from the recording plane, hence ideal reconstruction with the original parameters would have reconstructed the image at 13 in from the plate and with zero twist. This is reflected in column 3 of table 2, where the result is shown for a reconstruction with a collimated beam at 30 degrees (the original reference). The results also show that a small variation of the beam divergence at the correct angle shows very little twist; the amount of twist on the 30 degree row in table 2 is zero, or close to it. However, when the beam direction changes by +/- 10 degrees, there is significant twist and displacement, in the z direction, of the image. In the case of reconstruction at 633, the correct parameters for reconstruction are collimated at 40 degrees. It can be seen from table 3 that at this angle and at zero divergence, there is little twist but a significant displacement of the image in the z direction. Hence, if the image is reconstructed with a different angle than that at which it was exposed then there is a displacement away from the ideal image plane, despite being reconstructed at the correct angle as per the grating equation.

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[2] "Nonparaxial Imaging, Magnification, and Aberration Properties in Holography", Edwin B. Champagne, JOSA 1967, vol 57, 1, pp 51 – 55

[3] D. Gabor, Proc. Roy. Soc. (London) A197, 454 (1949)